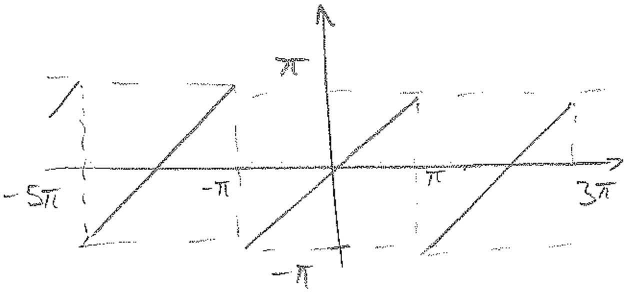
HW4 #1

$f : [0,\pi] \to \mathbb{R}$

$f(x) = x$

a) Sine Series. $\tilde{f}_{odd} : [-\pi,\pi] \to \mathbb{R}$ $\rightsquigarrow$ $\tilde{f} : \mathbb{R} \to \mathbb{R}$ periodic extension w/ period 2π

$\tilde{f}_{odd}(x) = x$



$$\tilde{f} \sim \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin(nx)\,dx$$

Int by parts

$$\stackrel{\text{Int by parts}}{=} \frac{2}{\pi}\left(- \frac{x\cos(nx)}{n}\Big|_0^{\pi} + \overset{0}{\cancel{\int_0^{\pi} \frac{\cos(nx)}{n}\,dx}} \right)$$

$$= \frac{2}{\pi}\left(- \frac{\pi\cos(n\pi)}{n} \right) = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

So $$\tilde{f}(x) \sim \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin(nx)$$

Note: By Fourier's Theorem, we have pointwise convergence of this series for all $x \in (-\pi,\pi)$, but not at $x = \pm\pi$. Thus, we may write

$$\boxed{f(x) = x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) \qquad \text{if} \quad 0 \le x < \pi}$$

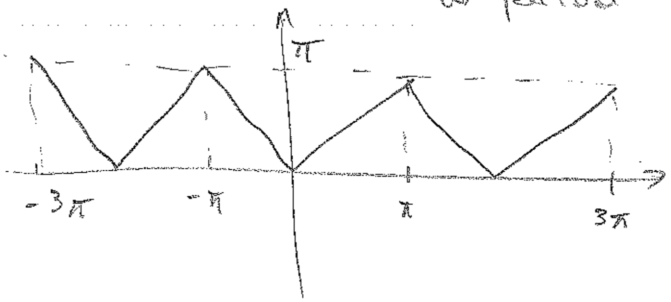
(and also if $-\pi < x < \pi$).

b) Cosine series

$\tilde{f}_{even}: [-\pi, \pi] \to \mathbb{R}$ $\rightsquigarrow$ $\tilde{f}: \mathbb{R} \to \mathbb{R}$

$\tilde{f}_{even}(x) = |x|$

periodic extension w/ period 2π



$$\tilde{f} \sim \sum_{n=0}^{\infty} a_n \cos(nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x \, dx = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx \overset{\text{Int. by parts}}{=} \frac{2}{\pi}\left(x\,\frac{\sin(nx)}{n}\Big|_0^{\pi} - \int_0^{\pi} \frac{\sin(nx)}{n}\,dx \right)$$

$$= \frac{2}{\pi}\left(\cancelto{0}{\frac{\pi\sin(n\pi)}{n}} + \frac{\cos(nx)}{n^2}\Big|_0^{\pi} \right) = \frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2} \right) = \frac{2}{\pi n^2}\left((-1)^n - 1\right).$$

Note that $a_n = 0$ if n is even, So, replacing $n = 2k-1$,

$$a_k = \frac{2(-1-1)}{\pi(2k-1)^2} = -\frac{4}{\pi(2k-1)^2}$$

Hence

$$\tilde{f}(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2k-1)^2}\cos((2k-1)x).$$

Note: By Fourier's theorem, we have pointwise convergence $\forall x \in \mathbb{R}$. Thus, we may write

$$\boxed{f(x) = x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2k-1)^2}\cos((2k-1)x), \text{ if } 0 \le x \le \pi}$$

(and also $\forall x \in \mathbb{R}$)

Cool fact: check out what happens if $x = \pi$...

$$\frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

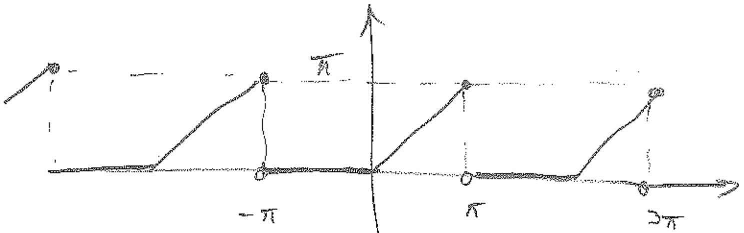
c) Fourier Series (w/ both sin and cos)

$\tilde{f} : [-\pi, \pi] \longrightarrow \mathbb{R}$ $\rightsquigarrow$ $\tilde{f}_{per.} : \mathbb{R} \longrightarrow \mathbb{R}$

periodic extension of period 2π

$$\tilde{f}(x) = \begin{cases} x & \text{if } x \in [0, \pi] \\ 0 & \text{if } x \in [-\pi, 0] \end{cases}$$

(this is an arbitrary choice of extension
there are many options...)



$$\tilde{f}_{per}(x) \sim \sum_{n=0}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

(Note: $\int_{-\pi}^{\pi} \tilde{f}(x)dx = \int_{0}^{\pi} \tilde{f}(x)dx$
b/c $\tilde{f}|_{[-\pi,0]} \equiv 0$.)

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} x\,dx = \frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x \cos nx \, dx = \frac{1}{\pi n^2}\left((-1)^n - 1\right) \quad \text{by b)}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x \sin nx \, dx = \frac{(-1)^{n+1}}{n} \quad \text{by a)}$$

Hence

$$\tilde{f}_{per.}(x) \sim \frac{\pi}{4} - \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}\cos((2k-1)x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx)$$

Note: By Fourier's Theorem, we have pointwise convergence if $x \in [0, \pi)$
Thus, we may write

$$\boxed{f(x) = x = \frac{\pi}{4} - \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}\cos((2k-1)x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx) \quad \text{if } 0 \le x < \pi}$$

This expression is precisely the average of the series in a) and b).

d) Pointwise convergence was studied in each item.

a) Fourier series equals $f(x)=x$ if $0 \le x < \pi$,

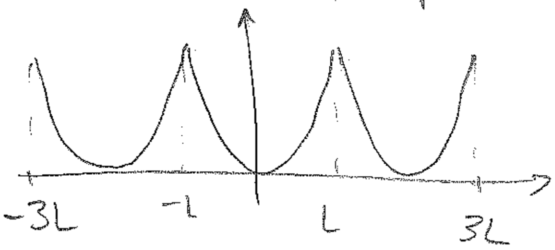
b) Fourier series equals $f(x)=x$ if $0 \le x \le \pi$,

c) Fourier series equals $f(x)=x$ if $0 \le x < \pi$,

#2

a) $g:[-L,L]\to\mathbb{R}$, $g(x)=x^2$ $\rightsquigarrow$ $\tilde{g}:\mathbb{R}\to\mathbb{R}$ periodic extension w/ period $2L$

$$\tilde{g}(x) \sim \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$



$$a_0 = \frac{1}{2L}\int_{-L}^{L} x^2\,dx = \frac{1}{2L}\,\frac{2L^3}{3} = \frac{L^2}{3}.$$

$$a_n = \frac{1}{L}\int_{-L}^{L} x^2\cos\left(\frac{n\pi x}{L}\right)dx \doteq \frac{2}{L}\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)dx \overset{y=\frac{n\pi x}{L}}{=} \frac{2L^2}{n^3\pi^3}\int_0^{n\pi} y^2\cos y\,dy$$

$$\overset{\text{Int. by parts}}{=} -\frac{4L^2}{n^3\pi^3}\int_0^{n\pi} y\sin y\,dy \overset{\text{Same integral as in \#1 a)}}{=} -\frac{4L^2}{n^3\pi^3}\left(-n\pi(-1)^n\right) = \frac{4L^2}{n^2\pi^2}(-1)^n$$

So

$$\tilde{g}(x) \sim \frac{L^2}{3} + \sum_{n=1}^{\infty}\frac{4L^2}{n^2\pi^2}(-1)^n\cos\left(\frac{n\pi x}{L}\right).$$

By Fourier's theorem, we have pointwise convergence of this series for all $x \in \mathbb{R}$. Thus, we may write

$$\boxed{g(x) = x^2 = \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{L}\right) . \text{ if } -L \leq x \leq L \quad \text{(and also for all } x \in \mathbb{R}\text{)}}$$

b) Applying the above to $x = L$, we get

$$L^2 = \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi)$$

$$\Rightarrow \quad \frac{2}{3} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n$$

$$\Rightarrow \quad \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

c) Instead of computing the Fourier Series for $h(x) = x^4$, we can integrate the series for $g(x) = x^2$ term by term twice, since all functions involved are piecewise C^1

$$x^2 = \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{L}$$

$$F(x) = x^2 - \frac{L^2}{3} = \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{L}$$

$$\int F(x)dx = \frac{x^3}{3} - \frac{L^2 x}{3} = \frac{1}{2L} \cancelto{0}{\int_{-L}^{L} F(x)dx} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \frac{L}{n\pi} \sin \frac{n\pi x}{L}$$

$$\underbrace{\frac{x^3}{3}-\frac{L^2x}{3}}_{G(x)} = \frac{4L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\left(\frac{n\pi x}{L}\right)$$

$$\int G(x)\,dx = \frac{x^4}{12}-\frac{L^2x^2}{6} = \frac{1}{2L}\int_{-L}^{L}G(x)\,dx + \frac{4L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\frac{L}{n\pi}\left(-\cos\frac{n\pi x}{L}\right)$$

$$= -\frac{7L^4}{180}+\frac{4L^4}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}\cos\frac{n\pi x}{L}$$

Applying at $x=L$, we get

$$\frac{L^4}{12}-\frac{L^4}{6} = -\frac{7L^4}{180}+\frac{4L^4}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}\underbrace{\cos(n\pi)}_{(-1)^n}$$

$$-\frac{1}{12} = -\frac{7}{180}-\frac{4}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$+\frac{8}{180} = +\frac{4}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\boxed{\frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4} = 1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots}$$