HW 9

Sebastian Severin

3. D unit disk in $\mathbb{R}^2$

$$\begin{cases} \Delta u = 4 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases} \qquad |u(0,\theta)| < +\infty$$

a) Laplacian on D: eigenvalues: $\lambda_{nm} = z_{nm}^2$ (z_{nm}: mth zero of nth Bessel function J_n)
$n = 0, 1, 2, \ldots,$
$m = 1, 2, 3, \ldots$

eigenfunctions: $\varphi_{nm}(r,\theta) = J_n(z_{nm} r)\cos(n\theta)$,
$\psi_{nm}(r,\theta) = J_n(z_{nm} r)\sin(n\theta)$

let p and q be determined by
$(nm) \longmapsto p, q$
$(01) \longmapsto 1$
$(11) \longmapsto 2$
$(12) \longmapsto 3$
$(22) \longmapsto 4$
$\vdots \qquad \vdots$

expand u and $f :\equiv 4$ in terms of eigenfunctions φ_p and ψ_p:

$$4 = f(r,\theta) = \sum_{p=1}^{\infty} A_p \varphi_p + \sum_{q=1}^{\infty} B_q \psi_q,$$

$$u(r,\theta) = \sum_{p=1}^{\infty} a_p \varphi_p + \sum_{q=1}^{\infty} b_q \psi_q$$

$$\Rightarrow \Delta u(r,\theta) = \sum_{p=1}^{\infty} a_p \Delta\varphi_p + \sum_{q=1}^{\infty} b_q \Delta\psi_q = -\sum_{p=1}^{\infty} a_p \lambda_p \varphi_p - \sum_{q=1}^{\infty} b_q \lambda_q \psi_q$$

$\Delta u = f$ holds $\Leftrightarrow$

$$0 = \sum_{p=1}^{\infty} (A_p + \lambda_p a_p)\varphi_p + \sum_{q=1}^{\infty} (B_q + \lambda_q b_q)\psi_q$$

$\lambda > 0$ b/c Dirichlet BC $\Leftrightarrow$ ($\uparrow$ $\{\varphi_q, \psi_q\}$ orthogonal set)

$$a_p = -\frac{A_p}{\lambda_p}, \quad b_q = -\frac{B_q}{\lambda_q} \qquad (*)$$

we know $\lambda_p = z_p^2$, $\lambda_q = z_q^2$,
$$A_p = \frac{\int_0^1 \int_{-\pi}^{\pi} 4\varphi_p \, r \, d\theta \, dr}{\int_0^1 \int_{-\pi}^{\pi} \varphi_p^2 \, r \, d\theta \, dr}, \quad B_q = \frac{\int_0^1 \int_{-\pi}^{\pi} 4\psi_q \, r \, d\theta \, dr}{\int_0^1 \int_{-\pi}^{\pi} \psi_q^2 \, r \, d\theta \, dr}$$

so, by (*), we know a_p, b_q and thus $u(r,\theta)$

~~b) you should get something like $u(r) = 1 - r^2$ which (thus) is the same as the solution you obtained in a) because both solve the same problem~~

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(3.) b) $f = f(r,\theta) \equiv 4$ is radial, so in fact $f = f(r) \equiv 4$

Hence, $u(r,\theta)$ must be radial, i.e. $u = u(r)$

separate variables: $\underbrace{u(r,\theta)}_{=u(r)} = R(r)\,T(\theta) \Rightarrow$ can choose $T(\theta) \equiv 1$

Now: $4 = \underbrace{\Delta u(r,\theta)}_{=u(r)} = \frac{1}{r}(rR')' + \frac{1}{r^2} R \underbrace{T''}_{\equiv 0 \text{ because } T(\theta)\equiv 1}$
$= \frac{1}{r}(rR')'$

$\Leftrightarrow (rR')' = 4r$

$\Rightarrow rR' = 2r^2 + c_0$ w/ $c_0 = $ const.

$\Leftrightarrow R' = 2r + \frac{c_0}{r}$

$\Rightarrow R = r^2 + c_0 \ln r + c_1$ w/ $c_1 = $ const.

(all in D)

apply BCs: $|R(0)| < +\infty \Rightarrow c_0 = 0$

$0 = R(1) = 1 + c_1 \Rightarrow c_1 = -1$

($u = 0$ on ∂D translates into $0 = R(1)$)

So, $R(r) = r^2 - 1$

$\Rightarrow u(r) = r^2 - 1$

Check that the solution is correct:

$R(r) = r^2 - 1 \Rightarrow R'(r) = 2r$

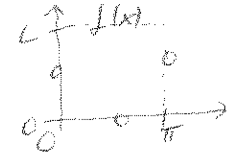
in D: $\Delta u = \frac{1}{r}(rR')' = \frac{1}{r}(2r^2)' = \frac{1}{r}(4r) = 4$ ✓

on ∂D: $u(1) = R(1) = 1 - 1 = 0$ ✓

The series obtained in a) is the Fourier-Bessel series of $u(r) = r^2 - 1$ because both solve the same problem.

HW9

(5) Haberman 8.6.6. $\Delta u = e^{2y} \sin x$



$u(0,y)=0 \quad u(x,0)=0$

$u(\pi,y)=0 \quad u(x,L)=f(x)$

$$u(x,y) = \sum_{n=1}^{\infty} a_n(y) \sin(nx)$$

$$\Rightarrow \Delta u(x,y) = \sum_{n=1}^{\infty} a_n''(y) \sin(nx) + \sum_{n=1}^{\infty} a_n(y)(-n^2)\sin(nx)$$

$\quad \| $

$e^{2y}\sin x$

$$\Rightarrow \begin{cases} a_n'' - n^2 a_n = 0 & , n \neq 1 \quad (1) \\ a_1'' - a_1 = e^{2y} & \quad (2) \end{cases}$$

(1) $a_n'' - n^2 a_n = 0 \Rightarrow a_n(y) = \tilde{\alpha}_n e^{ny} + \tilde{\beta}_n e^{-ny}$

$= \alpha_n \sinh(ny) + \beta_n \cosh(ny)$

(2) $a_1'' - a_1 = e^{2y}$

solution to homogeneous problem $a_1'' - a_1 = 0$ is $a_{1h}(y) = \alpha_1 \sinh(y) + \beta_1 \cosh(y)$ (see (1))

solve inhomogeneous problem by undetermined coefficients:

try $a_{1s}(y) := \alpha e^{2y}$, $\alpha = \text{const.} \in \mathbb{R}$.

$\Rightarrow a_{1s}''(y) = 4\alpha e^{2y}$

in (2) $\Rightarrow 4\alpha e^{2y} - \alpha e^{2y} = e^{2y} \iff (4-1)\alpha = 1 \iff \alpha = \frac{1}{3}$

So, $a_1 = a_{1s} + a_{1h} = \frac{1}{3}e^{2y} + \alpha_1 \sinh(y) + \beta_1 \cosh(y)$

Thus,

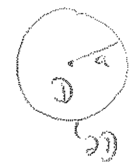
$u(x,y) = \sum_{n=1}^{\infty} a_n(y) \sin(nx)$ where $a_n(y) = \frac{1}{3} e^{2y} \delta_{n1} + \alpha_n \sinh(ny) + \beta_n \cosh(ny)$

$\delta_{n1} = \begin{cases} 1, & n=1 \\ 0, & n \neq 1 \end{cases}$

(See ~~also~~ solution in the back of the book)

HW 9

④ Haberman 8.6.3.



a) $\Delta u = Q(r,\theta)$, $u(a,\theta) = 0$

eigenfunctions of Laplace operator on disk of radius a:

$$\varphi_{nm}(r,\theta) = J_n(\sqrt{\lambda_{nm}}\, r)\cos(n\theta) ,$$

$$\psi_{nm}(r,\theta) = J_n(\sqrt{\lambda_{nm}}\, r)\sin(n\theta)$$

and eigenvalues of Laplace operator on disk of radius a:

$$\lambda_{nm} = \left(\frac{z_{nm}}{a}\right)^2$$ with z_{nm} ... mth zero of nth Bessel function J_n

note: $n = 0, 1, 2, \ldots,$
$m = 1, 2, 3, \ldots$

Expand u and Q in terms of φ_{nm}, ψ_{nm}:

$$u(r,\theta) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} A_{nm}\varphi_{nm} + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} B_{nm}\psi_{nm} ,$$

$$Q(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_{nm}\varphi_{nm} + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} b_{nm}\psi_{nm}$$

Now,

$$Q = \Delta u = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} A_{nm}\Delta\varphi_{nm} + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} B_{nm}\Delta\psi_{nm}$$

$$= -\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\lambda_{nm}(A_{nm}\varphi_{nm} + B_{nm}\psi_{nm})$$

$$\Rightarrow \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(\lambda_{nm}A_{nm} + a_{nm})\varphi_{nm} + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(\lambda_{nm}B_{nm} + b_{nm})\psi_{nm} = 0$$

$$\Rightarrow \begin{cases} \lambda_{nm}A_{nm} + a_{nm} = 0 \\ \lambda_{nm}B_{nm} + b_{nm} = 0 \end{cases} \Rightarrow \begin{cases} A_{nm} = -\dfrac{a_{nm}}{\lambda_{nm}} \\ B_{nm} = -\dfrac{b_{nm}}{\lambda_{nm}} \end{cases}$$

But we can compute a_{nm} and b_{nm} to get

$$A_{nm} = -\frac{a^2}{z_{nm}^2}\,\frac{\int_0^a\int_{-\pi}^{\pi} Q(r,\theta)J_n(\sqrt{\lambda_{nm}}\,r)\cos(n\theta)\,r\,dr\,d\theta}{\int_0^a J_n^2(\sqrt{\lambda_{nm}}\,r)\,r\,dr\int_{-\pi}^{\pi}\cos^2(n\theta)\,d\theta} ,$$

$$B_{nm} = -\frac{a^2}{z_{nm}^2}\,\frac{\int_0^a\int_{-\pi}^{\pi} Q(r,\theta)J_n(\sqrt{\lambda_{nm}}\,r)\sin(n\theta)\,r\,dr\,d\theta}{\int_0^a J_n^2(\sqrt{\lambda_{nm}}\,r)\,r\,dr\int_{-\pi}^{\pi}\sin^2(n\theta)\,d\theta}$$

There are always solutions.

5) $\Delta u = Q(r,\theta)$, $u_r(a,\theta)=0$

Same as a) but now z_{nm} = mth zero of J_n'

There are solutions if and only if $\int_D Q = 0$ (compare Problem ① of HW 9).

⑥ Haberman 8.6.7.

$\Delta u = Q(x,y,z)$, $x\in(0,L), y\in(0,H), z\in(0,W)$, $R := [0,L]\times[0,H]\times[0,W]$

$u \equiv 0$ on boundary

Eigenfunctions and eigenvalues of Laplace operator on R with Dirichlet BCs:

$$\left.\begin{aligned}\lambda_{nmp} &= \left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2+\left(\frac{p\pi}{W}\right)^2\\ \varphi_{nmp} &= \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\sin\left(\frac{p\pi z}{W}\right)\end{aligned}\right\} n,m,p=1,2,3,\ldots$$

Expand u, Q in terms of φ_{nmp} to get:

$$\underbrace{\Delta\Big(\underbrace{\textstyle\sum_{n,m,p=1}^{\infty} A_{nmp}\varphi_{nmp}}_{=u}\Big)}_{\parallel} = \sum_{n,m,p=1}^{\infty} A_{nmp}\Delta\varphi_{nmp} = -\sum_{n,m,p=1}^{\infty}\lambda_{nmp}A_{nmp}\varphi_{nmp}$$

$$Q(x,y,z) = \sum_{n,m,p=1}^{\infty} a_{nmp}\varphi_{nmp}$$

$$\Rightarrow \sum_{n,m,p=1}^{\infty}(\lambda_{nmp}A_{nmp}+a_{nmp})\varphi_{nmp}=0$$

$$\Rightarrow \lambda_{nmp}A_{nmp}+a_{nmp}=0 \quad \forall n,m,p\in\mathbb{N}$$

$$\Rightarrow A_{nmp} = -\frac{a_{nmp}}{\lambda_{nmp}} \quad \forall n,m,p\in\mathbb{N}$$

But we can compute a_{nmp}:

$$a_{nmp} = \frac{8}{LHW}\int_0^W\int_0^H\int_0^L Q(x,y,z)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\sin\left(\frac{p\pi z}{W}\right)dx\,dy\,dz$$

(7.) Prove: $g(x) = e^{ix\xi_0} f(x) \Rightarrow \hat{g}(\xi) = \hat{f}(\xi - \xi_0)$

Proof: $g(x) = e^{ix\xi_0} f(x)$

$$\Rightarrow \hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-i\xi x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix(\xi - \xi_0)} dx = \hat{f}(\xi - \xi_0)$$

□